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Pricing a contingent claim with random interval or fuzzy random payoff in one-period setting

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ABSTRACT

This article proposes a method for pricing a contingent claim with random interval and fuzzy random payoff. On introduction of the acceptability concept based on classical no-arbitrage argument, a price interval and a fuzzy price are obtained in random interval market and fuzzy random market, respectively. New definitions on replicative strategies, sub-replicative and sup-replicative ones, in two market setting are given. Some interesting results similar to those in the classical random market are presented.

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1. Introduction

In modern financial theories, prices of uncertain assets are modeled as random variables or stochastic processes. With the help of probability theory, we have seen great advances in financial analysis, including asset pricing, portfolio selection, risk management and consumption optimization, etc. If we model an uncertain quantity as a random variable, we should know all possible realizations of the quantity and the probability of the occurrence of every possible realization.

However, there are many settings where we can't model as random variables. For example [1], we do know the market is either "bullish" or "bearish", but we can't give precise values in two states. We should use other uncertain tools to target this imprecision. According to different properties of an uncertain quantity, it can be modeled as a random variable, an interval number, a fuzzy number or a fuzzy random variable [2]. In the example above, it seems more confident to view stock prices as two interval or fuzzy numbers. Fuzzy theory can be used to handle cases where vague or ambiguous information, such as "the price is about $20" and "the price will have a big jump", are involved.

With the exception of randomness in finance, uncertainties have motivated research in portfolio selection, such as [3–6] and asset pricing, such as [1,7–9]. We can see that different uncertain tools are used to model the security. A single-period setting model for contingent claim pricing is the simplest model in pricing theory[10–12]. Many important results and methods for pricing are extended from single period setting. All discussions are based on algebra and probability theory. This article will extend classical results to the market where all securities have random interval or fuzzy random payoffs.

This paper is organized as follows. In Section 2, we recall classical results and methods on contingent claim pricing theory in one-period setting. In Section 3, we will introduce the market with random interval payoffs. After proposing a concept

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of acceptable market based on no-arbitrage principle, we give the price interval for contingent claim with random interval payoffs. In Section 4, we will discuss the pricing problem of a contingent claim with fuzzy random payoffs. A procedure and an example will be given to explain the method. Some discussion and further development are finally given in Section 5.

2. Overview of classical contingent claim pricing theory in one-period setting

Throughout this article, we denote $Y = (y_1, \ldots, y_n) \geq 0$ for all $y_i \geq 0, i = 1, 2, \ldots, n$; $Y > 0$ for $Y \geq 0$ and $Y \neq 0$; $Y \gg 0$ for all $y_i > 0, i = 1, 2, \ldots, n$.

In one-period setting, there are two trading dates: date $t = 0$ and date $t = 1$. At $t = 0$, every security has its deterministic price quoted in the market. At $t = 1$, suppose there are M possible states, denoted as $\Omega = \{w_1, \ldots, w_M\}$. There are N basic securities (including stocks and bonds), whose prices at $t = 0$ are given as $S = (S_1, S_2, \ldots, S_N)^{\mathrm{T}}$. At date $t = 1$, any basic securities have random payoffs with state sets Ω. Because of the finiteness of possible state set, N basic securities have a payoff matrix $D = [D_{ij}]_{N\times M}$, where D_{ij} is the payoff of the ith security at state w_j. Denote the market composed of above N basic securities be $\mathcal{M} = (S, D)$.

$\theta = (\theta_1, \ldots, \theta_N)^{\mathrm{T}}$ is called a portfolio (trading strategy) of N basic securities, where θ_i is the unit amount of the ith security. As $\theta_i < 0$, it means the investor sells $-\theta_i$ units of the ith security short. Here we don't add any constraints on short-selling. Portfolio θ has its value $S^{\mathrm{T}}\theta$ at $t = 0$, and payoff vector $D^{\mathrm{T}}\theta$ at $t = 1$.

Definition 2.1. θ is called an arbitrage, if $S^{\mathrm{T}}\theta < 0, D^{\mathrm{T}}\theta \geq 0$ or $S^{\mathrm{T}}\theta \leq 0, D^{\mathrm{T}}\theta > 0$ hold at the same time.

Definition 2.2. A vector $\psi \gg 0$ is called a state price vector for the market $\mathcal{M}$, if $S = D\psi$.

Lemma ([12]). *$\mathcal{M}$ has no arbitrage opportunity if and only if the market has at least one state price vector.*

Now introduce a contingent claim whose price, h, at date 0 is to be determined with payoff given by $X = (X_1, \ldots, X_M)^{\mathrm{T}}$. We want to get the price for the claim X.

The price(price interval) for X will be got by no-arbitrage argument, which means the price h should be determined such that the introduction of X will not lead to arbitrage opportunities. h can be got from two ways: the first from state price vectors, and the second from replicative strategies.

Denote $\Psi = \{\psi \gg 0 | S = D\psi\}$ be all state price vectors in the market $\mathcal{M}$. From the lemma, we get $\Psi \neq \emptyset$ under no-arbitrage condition.

Definition 2.3. The market $\mathcal{M}$ is complete if any contingent claim X can be replicated by basic securities.

A complete market is an idealized market. In practice, we can't get any complete markets. Following theorem gives the relation between completeness and state price vectors.

Theorem 2.1. *$\mathcal{M}$ is complete, if and only if there is a unique state price vector.*

Set $h_- = \inf_{\psi\in\Psi} X^{\mathrm{T}}\psi,\ h_+ = \sup_{\psi\in\Psi} X^{\mathrm{T}}\psi$. Then the arbitrage-free price interval is $[h_-, h_+]$.

Proposition 2.1 ([11]). *Under no-arbitrage principle,*

(1) *if $h_- = h_+$, the claim X has a unique price given by h_- or h_+;*
(2) *if $h_- < h_+$, X has a price interval (h_-, h_+) which has two properties:*
   (a) *at any price level in the open interval, there exists no arbitrage opportunity;*
   (b) *at any price level out of the closed interval $[h_-, h_+]$, arbitrage opportunities exist.*

From above proposition, in a complete market, any contingent claim has a determined price. While in an incomplete market, only those replicable claims have unique prices. From no-arbitrage principle, claims that can't be replicated have a price interval with the property in above proposition.

Definition 2.4. (1) If θ satisfies $D^{\mathrm{T}}\theta \geq X$, we call θ a sup-replicative strategy for X. Denote all sup-replicative strategies for X be $\Theta_p(X) = \{\theta | D^{\mathrm{T}}\theta \geq X, \theta \in \mathbb{R}^N\}$.

(2) If θ satisfies $D^{\mathrm{T}}\theta \leq X$, we call θ a sub-replicative strategy for X. Denote all sub-replicative strategies for X be $\Theta_b(X) = \{\theta | D^{\mathrm{T}}\theta \leq X, \theta \in \mathbb{R}^N\}$.

By duality of linear programming for h_- and h_+, we can get following explanation for the price interval.

Proposition 2.2 ([11]).

$$h_- = \max_{\theta\in\Theta_b(X)} S^{\mathrm{T}}\theta, \qquad h_+ = \min_{\theta\in\Theta_p(X)} S^{\mathrm{T}}\theta.$$

The proposition above tells us, the minimum feasible price for X equals to the maximal value of sub-replicative portfolio of X; and the maximum feasible price for X equals to the minimal value of sup-replicative portfolio of X.

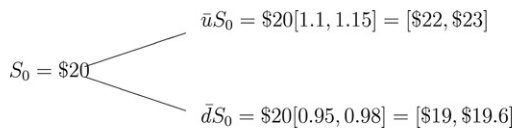

Fig. 1. A binomial tree with interval increasing/decreasing rate.

3. Acceptable prices for a contingent claim with random interval payoff

3.1. Basics about interval numbers

Definition 3.1. A closed interval in $\mathbb{R}$, $[a^L, a^U]$, is called an interval number, denoted as $\bar{a} = [a^L, a^U]$. a^L, a^U are called the lower and upper bounds of $\bar{a}$, respectively. If $a^L = a^U = a$, then $\bar{a}$ is just a crisp number.

Denote I be all interval numbers in $\mathbb{R}$.

For two interval numbers $\bar{a} = [a^L, a^U]$ and $\bar{b} = [b^L, b^U]$, following arithmetic operations are defined.

(i) $\bar{a} + \bar{b} = [a^L, a^U] + [b^L, b^U] = [a^L + b^L, a^U + b^U]$;
(ii) $-\bar{a} = [-a^U, -a^L]$;
(iii) $\bar{a} - \bar{b} = \bar{a} + (-\bar{b}) = [a^L - b^U, a^U - b^L]$;
(iv) for a real number k, $k\bar{a}$ takes $[ka^L, ka^U]$ for $k \geq 0$, and $[ka^U, ka^L]$ for $k < 0$.

Definition 3.2. Let $\bar{a} = [a^L, a^U]$, $\bar{b} = [b^L, b^U]$ be two interval numbers in $\mathbb{R}$. A partial order for comparing two interval numbers is given as $\bar{a} \preceq \bar{b}$ if and only if $a^U \leq b^L$. It means that $\bar{a}$ is inferior to $\bar{b}$. We also define another partial order, $\succeq$, as $\bar{a} \succeq \bar{b}$ if and only if $\bar{b} \preceq \bar{a}$.

This rule for comparing two interval numbers is just the strict order $<$ proposed in [13]. Under this order, we can be sure that any possible realization of $\bar{a}$ is less than realization of $\bar{b}$.

Definition 3.3. An interval number $\bar{a} = [a^L, a^U]$ is non-negative if $\bar{a} \succeq 0$ or equivalently $a^L \geq 0$. $\bar{a}$ is positive strictly, denoted as $\bar{a} \succ 0$ if $a^L > 0$.

Definition 3.4 (*[14]*). Given a probability space $(\Omega, \mathcal{A}, P)$, $\bar{a}(w) = [a^L(w), a^U(w)]$ is a random interval defined in Ω, if $a^L(w), a^U(w)$ are random variables, and for any $w \in \Omega$, $a^L(w) \leq a^U(w)$.

If the state set Ω is finite, say $\Omega = \{w_1, \ldots, w_M\}$, we can represent the random interval $\bar{a}(w)$ as a vector with all entries interval numbers, $\bar{a} = (\bar{a}(w_1), \ldots, \bar{a}(w_M))$.

3.2. One-period market with random interval payoffs

In classical one-period setting theory, securities have random payoffs so that they can be combined into a payoff matrix. In real market, we can't tell how much payoff will be at every possible state. In this case, giving an interval payoff at every state seems plausible. Binomial tree model is the simplest model in one-period setting. In a binomial model, the stock will decrease to dS_0 or increase to uS_0 at date 1, where S_0 is the price of stock at date 0. An argument about binomial tree model is how we can tell u, d precisely beforehand. So it seems acceptable that we view u, d as interval numbers, which means we can forecast the interval of the price at every possible state beforehand. For example, at date 0, the price of the stock is \$20. At date 1, it can be estimated that the increasing degree lies in [10%, 15%], the decreasing degree lying in [−5%, −2%]. Then we can have a binomial tree model as shown in Fig. 1.

Now we give the market model where every basic security has random interval payoff at date 1. As before, there are N basic securities, whose prices are $S = (S_1, \ldots, S_N)^{\mathrm{T}}$ given at $t = 0$. Different to the classical model, any basic security has a random interval payoff at date 1. The state set Ω is finite, $|\Omega| = M$. All possible payoffs of N basic securities are combined into an interval payoff matrix $\mathbb{D} = [\bar{D}_{ij}]_{N\times M}$, where $\bar{D}_{ij} = [D^L_{ij}, D^U_{ij}]$ is the interval payoff at state w_j of the ith security. And the market composed of above N basic securities is denoted as $\bar{\mathcal{M}} = (S, \mathbb{D})$. Set the lower and upper bound of $\mathbb{D}$ be $\mathbf{D}^L$ and $\mathbf{D}^U$, respectively. Obviously, we have

$$\mathbf{D}^L = [D^L_{ij}]_{N\times M}, \qquad \mathbf{D}^U = [D^U_{ij}]_{N\times M}. \tag{1}$$

Denote

$$\mathcal{D} = \{D | D = [D_{ij}]_{N\times M}, D_{ij} \in [D^L_{ij}, D^U_{ij}]\} = \{D | \mathbf{D}^L \leq D \leq \mathbf{D}^U\} \tag{2}$$

be the set of any possible payoff matrices of N securities realized at date 1.

Assumption 3.1. The first basic security has positive interval payoffs at every state, i.e., for any $j = 1, 2, \ldots, M$, $\bar{D}_{1j} \succ 0$, or equivalently $D^L_{1j} > 0$.

We can view a bond as a security with positive interval payoff. Suppose the interest rate is estimated as [2%, 2.5%] and the current price is \$10. Then the payoff of the bond at every state is same, [10.2, 10.25], a strictly positive interval number.

In mathematical finance, no-arbitrage principle is the foundation for discussion. By no-arbitrage argument, a crisp market (S, D) is reasonable if S and D lead to no arbitrages. Note that even D is fixed, there are many vectors S such that (S, D) is reasonable. In the reference [1], a method is proposed from a new definition of no-arbitrage in a market with fuzzy payoff matrix. The argument in this paper is quite different from that in [1]. Now in interval payoff setting, every matrix $D \in \mathcal{D}$ is viewed as possible realization. Every D leads to some reasonable price vectors S. So we want to find all reasonable price vectors related to some possible realization payoff matrix. Which is to say, we don't give arbitrage definition in the interval payoff model. We will see the market with interval payoff as a collection of numerous crisp markets:

$$(S, \mathbb{D}) = \{(S, D) | D \in \mathcal{D}\}.$$

In contrast to the classical argument, in this paper we will discuss the pricing problem from the following concept of an *acceptable market*, which is different but derived from crisp no-arbitrage argument.

Definition 3.5. The market $\bar{\mathcal{M}} = (S, \mathbb{D})$ is an acceptable market if there exists a crisp payoff matrix $D \in \mathcal{D}$ such that the crisp market $\mathcal{M} = (S, D)$ has no arbitrage opportunities. If $(S, \mathbb{D})$ is acceptable, we call S acceptable price for $\mathbb{D}$.

In interval market version, because we don't know precise payoff matrix at date 1, and every matrix in $\mathcal{D}$ is possible, the concept of acceptable price admits us to get all reasonable prices.

Proposition 3.1. *The interval market $\bar{\mathcal{M}}$ is acceptable if only if there exists a $\psi \gg 0$, such that*

$$\mathbf{D}^L \psi \le S \le \mathbf{D}^U \psi. \tag{3}$$

Proof. Necessity. From the lemma in previous section, $\bar{\mathcal{M}}$ is acceptable, if and only if there exists a $D \in \mathcal{D}$ and a $\psi \gg 0$ such that $S = D\psi$. Note $\mathbf{D}^L \le D \le \mathbf{D}^U$ and $\psi_i > 0$, $i = 1, 2, \ldots, M$. We can see for the same ψ, $\mathbf{D}^L \psi \le S \le \mathbf{D}^U \psi$ holds.

Sufficiency. Suppose $\mathbf{D}^L \psi \le S \le \mathbf{D}^U \psi$ hold for some $\psi \gg 0$. Rewrite the inequality as

$$\begin{cases} \mathbf{D}^L_1 \psi \le S_1 \le \mathbf{D}^U_1 \psi \\ \mathbf{D}^L_2 \psi \le S_2 \le \mathbf{D}^U_2 \psi \\ \qquad \vdots \\ \mathbf{D}^L_N \psi \le S_N \le \mathbf{D}^U_N \psi \end{cases} \tag{4}$$

where $\mathbf{D}^L_i$ and $\mathbf{D}^U_i$ are the ith row in $\mathbf{D}^L$ and $\mathbf{D}^U$ respectively.

Then there exist N positive numbers $\lambda_i \in [0, 1]$, $i = 1, 2, \ldots, N$ such that

$$S_i = (1 - \lambda_i)\mathbf{D}^L_i \psi + \lambda_i \mathbf{D}^U_i \psi, \quad i = 1, 2, \ldots, N.$$

Setting

$$D = \begin{pmatrix} (1 - \lambda_1)\mathbf{D}^L_1 + \lambda_1 \mathbf{D}^U_1 \\ (1 - \lambda_2)\mathbf{D}^L_2 + \lambda_2 \mathbf{D}^U_2 \\ \vdots \\ (1 - \lambda_N)\mathbf{D}^L_N + \lambda_N \mathbf{D}^U_N \end{pmatrix} \in \mathcal{D} \tag{5}$$

gives $S = D\psi$.

Definition 3.6. Call any $\psi \gg 0$ satisfying $\mathbf{D}^L \psi \le S \le \mathbf{D}^U \psi$ an acceptable state price vector for acceptable market $\bar{\mathcal{M}}$. Denote

$$\Psi = \{\psi \gg 0 | \mathbf{D}^L \psi \le S \le \mathbf{D}^U \psi\} \tag{6}$$

be the set of all acceptable state price vectors.

Under the argument of acceptable market, the set of acceptable state price vectors is derived. From the perspective that the acceptable market is from the classical no-arbitrage argument, the concept of acceptable state price vector can be seen as a generalization of classical definition of state price vector. In [1], a fuzzy state price vector is obtained and then used to price any claim. As was said above, the market with interval payoff matrix is viewed as a collection of crisp markets. Because a crisp market has crisp state price vectors, from the view of acceptability, we don't need to get a interval-valued state price vector. All acceptable state price vectors are crisp. Also, because of the same reason, the market with interval-valued payoff is obviously incomplete. Then the completeness of the market will not be discussed.

Proposition 3.2. *Under Assumption 3.1, Ψ is a bounded convex set with the closure $\bar{\Psi} = \{\psi \geq 0 | \mathbf{D}^L \psi \leq S \leq \mathbf{D}^U \psi\}$.*

Proof. Consider the first inequality in the constraint of Ψ, we have for any $\psi \in \Psi$,

$$(\min_{j=1,\ldots,M} D^L_{1j}) \left(\sum_{i=1}^{M} \psi_i \right) \leq \sum_{j=1}^{M} D^L_{1j} \psi_j \leq S_1 \leq \sum_{j=1}^{M} D^U_{1j} \psi_j \leq (\max_{j=1,\ldots,M} D^U_{1j}) \left(\sum_{i=1}^{M} \psi_i \right)$$

Under Assumption 3.1, we have $\max_{j=1,\ldots,M} D^U_{1j} \geq \min_{j=1,\ldots,M} D^L_{1j} > 0$. Then we can get for any $\psi \in \Psi$,

$$\frac{S_1}{\max\limits_{j=1,\ldots,M} D^U_{1j}} \leq \sum_{j=1}^{M} \psi_j \leq \frac{S_1}{\min\limits_{j=1,\ldots,M} D^L_{1j}}. \tag{7}$$

For convexity, consider $\psi_1, \psi_2 \in \Psi$ and any $\lambda \in [0, 1]$. From $\mathbf{D}^L \psi_1 \leq S \leq \mathbf{D}^U \psi_1$ and $\mathbf{D}^L \psi_2 \leq S \leq \mathbf{D}^U \psi_2$, we can get

$$\mathbf{D}^L [\lambda \psi_1 + (1 - \lambda) \psi_2] \leq S \leq \mathbf{D}^U [\lambda \psi_1 + (1 - \lambda) \psi_2]$$

which confirms $\lambda \psi_1 + (1 - \lambda) \psi_2 \in \Psi$ combined with $\lambda \psi_1 + (1 - \lambda) \psi_2 \gg 0$.

Obviously, the closure of Ψ is $\bar{\Psi} = \{\psi \geq 0 | \mathbf{D}^L \psi \leq S \leq \mathbf{D}^U \psi\}$.

3.3. Acceptable prices for a contingent claim

Now we introduce a contingent claim with interval payoff at every state. Such contingent claim is also defined by its payoff vector at date 1, $\mathbb{X} = (\bar{X}_1, \ldots, \bar{X}_M)^{\mathrm{T}}$, where $\bar{X}_j = [\bar{X}^L_j, \bar{X}^U_j]$ is the interval payoff of the claim at state w_j. Set $\mathbf{X}^L = (\bar{X}^L_1, \ldots, \bar{X}^L_M)^{\mathrm{T}}$ and $\mathbf{X}^U = (\bar{X}^U_1, \ldots, \bar{X}^U_M)^{\mathrm{T}}$ be the minimal and maximal payoff vector at date $t = 1$. The price of such claim, denoted as h, is up to be determined. Denote the market $\bar{\mathcal{M}}$ combined with the contingent claim be $\hat{\mathcal{M}} = \left(\begin{pmatrix} S \\ h \end{pmatrix}, \begin{pmatrix} \mathbb{D} \\ \mathbb{X}^{\mathrm{T}} \end{pmatrix} \right)$.

Under acceptability consideration, h will be set such that the market $\hat{\mathcal{M}}$ is acceptable. From the proposition in previous section, $\hat{\mathcal{M}}$ is acceptable if and only if

$$\begin{pmatrix} \mathbf{D}^L \\ (\mathbf{X}^L)^{\mathrm{T}} \end{pmatrix} \psi \leq \begin{pmatrix} S \\ h \end{pmatrix} \leq \begin{pmatrix} \mathbf{D}^U \\ (\mathbf{X}^U)^{\mathrm{T}} \end{pmatrix} \psi \tag{8}$$

holds for some $\psi \gg 0$. Then there exists a $\psi \in \Psi$ such that following two expressions hold at the same time:

$$\mathbf{D}^L \psi \leq S \leq \mathbf{D}^U \psi, \tag{9}$$

$$(\mathbf{X}^L)^{\mathrm{T}} \psi \leq h \leq (\mathbf{X}^U)^{\mathrm{T}} \psi. \tag{10}$$

Eq. (9) tells us $\psi \in \Psi$. From (10), for any $\psi \in \Psi$, we can get an acceptable price interval $[(\mathbf{X}^L)^{\mathrm{T}} \psi, (\mathbf{X}^U)^{\mathrm{T}} \psi]$. As ψ takes all vectors in Ψ, we can get a price interval for $\mathbb{X}$,

$$AP(\mathbb{X}) = \left[\inf_{\psi \in \Psi} (\mathbf{X}^L)^{\mathrm{T}} \psi, \sup_{\psi \in \Psi} (\mathbf{X}^U)^{\mathrm{T}} \psi \right]. \tag{11}$$

Denote $h_1 = \inf_{\psi \in \Psi} (\mathbf{X}^L)^{\mathrm{T}} \psi$ and $h_2 = \sup_{\psi \in \Psi} (\mathbf{X}^U)^{\mathrm{T}} \psi$.

Proposition 3.3. (1) *$AP(\mathbb{X})$ is well-defined and bounded. All acceptable price for h should be included in $AP(\mathbb{X})$.*

(2) *For any price level $h \in (h_1, h_2)$, there exist a $D \in \mathcal{D}$ and an $\mathbf{X}^L \leq X \leq \mathbf{X}^U$ such that the market $\left(\begin{pmatrix} S \\ h \end{pmatrix}, \begin{pmatrix} D \\ X^{\mathrm{T}} \end{pmatrix} \right)$ has no arbitrages.*

Proof. (1) Noticing Ψ is a convex bounded set, h_1, h_2 are well defined.

Using (7), for any $\psi \in \Psi$, we have

$$\frac{S_1 \min\limits_j \mathbf{X}^{\mathbf{L}}_{\mathbf{j}}}{\max\limits_j D^U_{1j}} \leq (\mathbf{X}^L)^{\mathrm{T}} \psi \leq \frac{S_1 \max\limits_j \mathbf{X}^{\mathbf{L}}_{\mathbf{j}}}{\min\limits_j D^L_{1j}} \tag{12}$$

and

$$\frac{S_1 \min\limits_j \mathbf{X}^{\mathbf{U}}_{\mathbf{j}}}{\max\limits_j D^U_{1j}} \leq (\mathbf{X}^U)^{\mathrm{T}} \psi \leq \frac{S_1 \max\limits_j \mathbf{X}^{\mathbf{U}}_{\mathbf{j}}}{\min\limits_j D^L_{1j}} \tag{13}$$

which shows the boundness of $AP(\mathbb{X})$.

Suppose h_0 be an acceptable price for $\mathbb{X}$. There exists a $\psi' \in \Psi$ and an $\mathbf{X}^L \le X \le \mathbf{X}^U$ such that $h_0 = X^{\mathrm{T}}\psi'$. Obviously, we have $h_1 \le h_0 \le h_2$ obviously.

(2) Note that we have

$$\inf_{\psi\in\Psi}(\mathbf{X}^L)^{\mathrm{T}}\psi \le \sup_{\psi\in\Psi}(\mathbf{X}^L)^{\mathrm{T}}\psi \le \inf_{\psi\in\Psi}(\mathbf{X}^U)^{\mathrm{T}}\psi \le \sup_{\psi\in\Psi}(\mathbf{X}^U)^{\mathrm{T}}\psi. \tag{14}$$

For any level $h \in (h_1, h_2)$, we prove the result in three cases:

(i) For $\inf_{\psi\in\Psi}(\mathbf{X}^L)^{\mathrm{T}}\psi < h < \sup_{\psi\in\Psi}(\mathbf{X}^L)^{\mathrm{T}}\psi$, from the convexity of Ψ, we can find a $\psi_0 \in \Psi$ such that $h = (\mathbf{X}^L)^{\mathrm{T}}\psi_0$. And from the acceptability of $(S, \mathbb{D})$, we can find a $D_0 \in \mathcal{D}$ such that $S = D_0\psi_0$. So ψ_0 is a state price vector of the market $\left(\begin{pmatrix} S \\ h \end{pmatrix}, \begin{pmatrix} D_0 \\ (\mathbf{X}^L)^{\mathrm{T}} \end{pmatrix}\right)$, which means the market has no arbitrage.

(ii) For $\inf_{\psi\in\Psi}(\mathbf{X}^U)^{\mathrm{T}}\psi < h < \sup_{\psi\in\Psi}(\mathbf{X}^U)^{\mathrm{T}}\psi$, the same argument as in the case (i) gives a no-arbitrage market.

(iii) For $\sup_{\psi\in\Psi}(\mathbf{X}^L)^{\mathrm{T}}\psi \le h \le \inf_{\psi\in\Psi}(\mathbf{X}^U)^{\mathrm{T}}\psi$, we have $(\mathbf{X}^L)^{\mathrm{T}}\psi_1 < h < (\mathbf{X}^U)^{\mathrm{T}}\psi_1$ for any $\psi_1 \in \Psi$. We can find a $\lambda \in [0, 1]$ such that

$$h = [\lambda\mathbf{X}^L + (1-\lambda)\mathbf{X}^U]^{\mathrm{T}}\psi_1.$$

Obviously, there exists a $D_1 \in \mathcal{D}$ such that $S = D_1\psi_1$. So the market

$$\left(\begin{pmatrix} S \\ h \end{pmatrix}, \begin{pmatrix} D_1 \\ (\lambda\mathbf{X}^L + (1-\lambda)\mathbf{X}^U)^{\mathrm{T}} \end{pmatrix}\right)$$

has no arbitrage.

Define two linear programming (I) and (II) as follows.

$$\text{(I)}\ \begin{array}{cc} \min & (\mathbf{X}^L)^{\mathrm{T}}\psi \\ \text{s.t.} & \mathbf{D}^L\psi \le S \\ & S \le \mathbf{D}^U\psi \\ & \psi \ge 0 \end{array} \qquad \text{(II)}\ \begin{array}{cc} \max & (\mathbf{X}^U)^{\mathrm{T}}\psi \\ \text{s.t.} & \mathbf{D}^L\psi \le S \\ & S \le \mathbf{D}^U\psi \\ & \psi \ge 0. \end{array} \tag{15}$$

Obviously, h_1, h_2 are optimal values of (I) and (II), respectively.

Definition 3.7. (1) θ is called a sup-replicative strategy for $\mathbb{X}$, if $\mathbb{D}^{\mathrm{T}}\theta \succeq \mathbb{X}$, or equivalently, $\mathbf{X}^U \le (\mathbb{D}^{\mathrm{T}}\theta)^L$. Denote all sup-replicative strategies of $\mathbb{X}$ as $\Theta_p(\mathbb{X})$.

(2) θ is called a sub-replicative strategy, if $\mathbb{X} \succeq \mathbb{D}^{\mathrm{T}}\theta$, or equivalently, $(\mathbb{D}^{\mathrm{T}}\theta)^U \le \mathbf{X}^L$. Denote all sub-replicative strategies of $\mathbb{X}$ as $\Theta_b(\mathbb{X})$.

Above definitions about sup/sub-replicative strategy are somewhat strict. $\mathbb{D}^{\mathrm{T}}\theta \succeq \mathbb{X}$ makes θ a sup-replicative strategy in a market with deterministic payoffs for any realizations of basic securities, D, and the contingent claim, X. The same argument holds for sub-replicative strategy.

Now introduce another two linear programming (III) and (IV).

$$\text{(III)}\ \begin{array}{cc} \max & S^{\mathrm{T}}\theta \\ \text{s.t.} & \mathbb{D}^{\mathrm{T}}\theta \preceq \mathbb{X} \\ & \theta \in \mathbb{R}^N \end{array} \qquad \text{(IV)}\ \begin{array}{cc} \min & S^{\mathrm{T}}\theta \\ \text{s.t.} & \mathbb{X} \preceq \mathbb{D}^{\mathrm{T}}\theta \\ & \theta \in \mathbb{R}^N. \end{array} \tag{16}$$

From the structure of programming (III) and (IV), we can interpret the optimal value of (III) be the maximal value of sub-replicative strategies for X, and the optimal value of (IV) be the minimal value of sup-replicative strategies for X.

Proposition 3.4. (III) *is the dual of* (I)*, and* (IV) *is the dual of* (II).

Proof. The dual of (I) is

$$\text{(DI)}\ \begin{array}{cc} \max & S^{\mathrm{T}}\theta_1 - S^{\mathrm{T}}\theta_2 \\ \text{s.t.} & (\mathbf{D}^U)^{\mathrm{T}}\theta_1 - (\mathbf{D}^L)^{\mathrm{T}}\theta_2 \le \mathbf{X}^L \\ & \theta_1, \theta_2 \ge 0. \end{array}$$

Note that (III) is equivalent to following programming (III$'$):

$$\text{(III}'\text{)}\ \begin{array}{cc} \max & S^{\mathrm{T}}\theta \\ \text{s.t.} & (\mathbb{D}^{\mathrm{T}}\theta)^U \le \mathbf{X}^L \\ & \theta \in \mathbb{R}^N. \end{array}$$

Setting $\theta = \theta_1 - \theta_2$, $\theta_1, \theta_2 \ge 0$, we can write (III$'$) as

$$\text{(III}''\text{)}\ \begin{array}{cc} \max & S^{\mathrm{T}}(\theta_1 - \theta_2) \\ \text{s.t.} & (\mathbb{D}^{\mathrm{T}}(\theta_1 - \theta_2))^U \le \mathbf{X}^L \\ & \theta_1, \theta_2 \ge 0. \end{array}$$

Note that under the positive constraints of θ_1, θ_2,

$$(\mathbb{D}^{\mathrm{T}}(\theta_1 - \theta_2))^U = (\mathbb{D}^{\mathrm{T}}\theta_1)^U - (\mathbb{D}^{\mathrm{T}}\theta_2)^L = (\mathbf{D}^U)^{\mathrm{T}}\theta_1 - (\mathbf{D}^L)^{\mathrm{T}}\theta_2. \tag{17}$$

So (III″) is just (DI), and then (III) is the dual of (I).

Similarly, we can prove (IV) is also the dual of (II).

According to dual theorems, h_1, h_2 are optimal values of (III) and (IV) respectively. We get following result similar to that in classical market.

Proposition 3.5.

$$h_1 = \max_{\theta\in\Theta_b(\mathbb{X})} S^{\mathrm{T}}\theta, \qquad h_+ = \min_{\theta\in\Theta_p(\mathbb{X})} S^{\mathrm{T}}\theta. \tag{18}$$

Then we can get the same results as those in the classic market:
(1) *the minimal acceptable price of a contingent claim is the maximal value of sub-replicative strategies for the claim;*
(2) *the maximal acceptable price of a contingent claim is the minimal value of sup-replicative strategies for the claim.*

Remark. The classical market with random payoffs is the special setting of the market with random interval payoffs. Note that if $\mathbf{D}^L = \mathbf{D}^U = \mathbf{D}$ and $\mathbf{X}^L = \mathbf{X}^U = \mathbf{X}$, the acceptable state price set is just $\Psi = \{\psi \gg 0 | S = \mathbf{D}\psi\}$, the classical state price set in Section 2.

Also we have the price interval be $[h_1, h_2]$, where $h_1 = \inf_{\psi\in\Psi}(\mathbf{X}^L)^{\mathrm{T}}\psi = \inf_{\psi\in\Psi}\mathbf{X}^{\mathrm{T}}\psi$ and $h_2 = \sup_{\psi\in\Psi}(\mathbf{X}^U)^{\mathrm{T}}\psi = \sup_{\psi\in\Psi}\mathbf{X}^{\mathrm{T}}\psi$ are the same endpoints as h_- and h_+ in Section 2. And the concepts of sub-replicative and sup-replicative strategies are as same as those in Section 2. The result in Proposition 3.5 is the same as that in Proposition 2.2.

3.4. An illustrative example

Suppose there be two basic securities S_1, S_2, whose prices are given by the vector $S = (2, 5)^{\mathrm{T}}$ at date 0. At date 1, three possible states may happen, and the payoff matrix is given by $\mathbb{D} = \begin{bmatrix} [1, \frac{3}{2}] & [\frac{5}{4}, \frac{5}{2}] & [\frac{9}{4}, 3] \\ [3, \frac{7}{2}] & [4, \frac{9}{2}] & [\frac{17}{4}, 6] \end{bmatrix}$.

Note that $\mathbf{D}^L = \begin{bmatrix} 1 & \frac{5}{4} & \frac{9}{4} \\ 3 & 4 & \frac{17}{4} \end{bmatrix}$ and $\mathbf{D}^U = \begin{bmatrix} \frac{3}{2} & \frac{5}{2} & 3 \\ \frac{7}{2} & \frac{9}{2} & 6 \end{bmatrix}$.

It can be checked $\psi = (1, \frac{1}{4}, \frac{1}{8})^{\mathrm{T}}$ is a positive vector such that $\mathbf{D}^L\psi \le S \le \mathbf{D}^U\psi$, which tells us the market $(S, \mathbb{D})$ is acceptable.

Now introduce a claim X with payoff $X = ([2, \frac{7}{2}], [3, \frac{13}{4}], [5, \frac{13}{2}])$. Under acceptability consideration, we can get the price interval of X be $[h^L, h^U]$, where h^L and h^U are the optimal values of following two programming, respectively.

$$\begin{aligned} h^L = \min \quad & 2\psi_1 + 3\psi_2 + 5\psi_3 \\ \text{s.t.} \quad & \psi_1 + \frac{5}{4}\psi_2 + \frac{9}{4} \le 2 \\ & 3\psi_1 + 4\psi_2 + \frac{17}{4}\psi_3 \le 5 \\ & \frac{3}{2}\psi_1 + \frac{5}{2}\psi_2 + 3\psi_3 \ge 2 \\ & \frac{7}{2}\psi_1 + \frac{9}{2}\psi_2 + 6\psi_3 \ge 5 \\ & \psi_1, \psi_2, \psi_3 \ge 0 \end{aligned} \qquad \begin{aligned} h^U = \max \quad & \frac{7}{2}\psi_1 + \frac{13}{4}\psi_2 + \frac{13}{2}\psi_3 \\ \text{s.t.} \quad & \psi_1 + \frac{5}{4}\psi_2 + \frac{9}{4} \le 2 \\ & 3\psi_1 + 4\psi_2 + \frac{17}{4}\psi_3 \le 5 \\ & \frac{3}{2}\psi_1 + \frac{5}{2}\psi_2 + 3\psi_3 \ge 2 \\ & \frac{7}{2}\psi_1 + \frac{9}{2}\psi_2 + 6\psi_3 \ge 5 \\ & \psi_1, \psi_2, \psi_3 \ge 0. \end{aligned}$$

It can be solved easily to get $h^L = 20/7$, $h^U = 129/20$.

4. Acceptable prices in the market with fuzzy random payoffs

4.1. Basics about fuzzy random variable

A fuzzy set in the real line $\mathbb{R}$, denoted as $\tilde{a}$, is defined by its membership function $\mu_{\tilde{a}}(x) : \mathbb{R} \to [0, 1]$. The α-level set of $\tilde{a}$, denoted as $\tilde{a}_\alpha$ is defined by $\tilde{a}_\alpha = \{x | \mu_{\tilde{a}}(x) \ge \alpha\}$ for $0 < \alpha \le 1$. And 0-level set of $\tilde{a}$, $\tilde{a}_0$ is defined by the closure of $\{x | \mu_{\tilde{a}}(x) > 0\}$.

Definition 4.1. A fuzzy set $\tilde{a}$ in $\mathbb{R}$ is called a fuzzy number, if following statements are satisfied:

(i) $\tilde{a}$ is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $\mu_{\tilde{a}}(x_0) = 1$.

(ii) $\mu_{\tilde{a}}(x)$ is quasi-concave, i.e., $\mu_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min(\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y))$ for any $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}$.
(iii) $\mu_{\tilde{a}}(x)$ is upper semicontinuous, i.e., for any $\alpha \in [0, 1]$, the α-level set of $\tilde{a}$ is closed.
(iv) 0-level set of $\tilde{a}$ is compact.

From the definition of a fuzzy number, for any $\alpha \in [0, 1]$, the α-level set of $\tilde{a}$ is a closed interval, denoted as $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$. Following representation theorem set an one-to-one map from a fuzzy number to its α-level sets.

A crisp number, b, in $\mathbb{R}$ can be viewed as a special fuzzy number whose membership function takes 1 at b, 0 elsewhere. An interval number in $\mathbb{R}$ can also be seen as a fuzzy number with its membership function taking 1 in I, 0 elsewhere. The set of all fuzzy numbers in $\mathbb{R}$ is denoted as $\mathcal{F}(\mathbb{R})$.

Theorem 4.1 (*[15]*). *Let $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ be the α-level set of fuzzy number $\tilde{a}$, $\alpha \in [0, 1]$. Then $\tilde{a}^L_\alpha, \tilde{a}^U_\alpha$ can be regarded as functions on* $[0, 1]$, *which satisfy*

(i) *$\tilde{a}^L_\alpha$ is nondecreasing and left-continuous*
(ii) *$\tilde{a}^U_\alpha$ is nonincreasing and left-continuous*
(iii) $\tilde{a}^L_1 \leq \tilde{a}^U_1$
(iv) *$\tilde{a}^L_\alpha, \tilde{a}^U_\alpha$ are right-continuous at $\alpha = 0$.*

Conversely, for any function $u(\alpha)$, $v(\alpha)$ defined on $[0, 1]$ which satisfy above four statements (i)–(iv), there exists a unique fuzzy number $\tilde{a}$ such that for all $\alpha \in [0, 1]$, $\tilde{a}_\alpha = [u(\alpha), v(\alpha)]$.

Definition 4.2. For two fuzzy number $\tilde{a}, \tilde{b}$ in $\mathbb{R}$, define a partial order $\prec$ as
$\tilde{a} \prec \tilde{b}$, if and only if $\tilde{a}_\alpha \preceq \tilde{b}_\alpha$ holds for any $\alpha \in [0, 1]$.
Using the partial order $\preceq$ for two interval numbers, we can get the equivalent definition for $\prec$ as
$\tilde{a} \prec \tilde{b}$, if and only if $\tilde{a}^U_\alpha \leq \tilde{b}^L_\alpha$ for any $\alpha \in [0, 1]$.

Fuzzy random variables have been considered in the setting of a random experiment to model an essentially mechanism associating a fuzzy value with each experimental outcome [16]. On the concept of fuzzy random variable, different definitions have been given. Here we use the definition by Puri and Ralescu [17].

Definition 4.3. Let $(\Omega, \mathcal{A}, P)$ be a probability space. A mapping $\tilde{a}(w) : \Omega \to \mathcal{F}(\mathbb{R})$ is called a fuzzy random variable on $(\Omega, \mathcal{A})$, if for any $\alpha \in [0, 1]$,

$$\tilde{a}_\alpha(w) = \{x | x \in \mathbb{R}, \mu_{\tilde{a}(w)}(x) \geq \alpha\} = [\tilde{a}^L_\alpha(w), \tilde{a}^U_\alpha(w)]$$

is a random interval, that is $\tilde{a}^L_\alpha(w), \tilde{a}^U_\alpha(w)$ are two random variables on $(\Omega, \mathcal{A})$.

Note that in a finite state space, say $\Omega = \{w_1, \ldots, w_M\}$, a fuzzy random variable $\tilde{a}(w)$ can be represented as a fuzzy number vector, $\tilde{a}_w = (\tilde{a}(w_1), \ldots, \tilde{a}(w_M))$.

Denote the set of all fuzzy random variables in $(\Omega, \mathcal{A})$ be $\mathcal{FR}(\Omega)$. And define the partial relation between two fuzzy random variables $\tilde{a}_w, \tilde{b}_w$ as
$\tilde{a}_w \leq \tilde{b}_w$ if and only if $\tilde{a}(w) \prec \tilde{b}(w)$ for any $w \in \Omega$.

Combined $\preceq$ for interval numbers and $\prec$ for fuzzy numbers, we can get a simple rule for comparing two fuzzy random variables.

Proposition 4.1. *$\tilde{a}_w \leq \tilde{b}_w$ if and only if for any $\alpha \in [0, 1]$, and any $w \in \Omega$, $\tilde{a}^U_\alpha(w) \leq \tilde{b}^L_\alpha(w)$.*

4.2. A market with fuzzy random payoffs

In previous section, we have extended the market with random payoffs to random interval payoffs. We also can argue that in many settings, only an ambiguous payoff can be told. For example, we can forecast that a stock's price will go up to about \$15, or go down to about \$10. In the statement, two ambiguous numbers are used. We can model "about \$15" and "about \$10" as two fuzzy numbers, and get a fuzzy random payoff for this stock. So this leads to the discussion on pricing contingent claims with fuzzy random payoffs.

Suppose at date 1, there are M possible states, with the same state set as before, denoted by $\Omega = \{w_1, \ldots, w_M\}$.$N$ basic securities have fuzzy random payoffs at date 1. So we can model N basic securities by their price vector $S = (S_1, \ldots, S_N)^{\mathrm{T}}$ and their fuzzy payoff matrix

$$\tilde{\mathbb{D}} = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} & \cdots & \tilde{D}_{1M} \\ \tilde{D}_{21} & \tilde{D}_{22} & \cdots & \tilde{D}_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ \tilde{D}_{N1} & \tilde{D}_{N2} & \cdots & \tilde{D}_{NM} \end{bmatrix}$$

where $\tilde{D}_{ij}$ is the fuzzy payoff of security S_i at state w_j. Denote the market composed of N basic securities as $\tilde{\mathcal{M}} = (S, \tilde{\mathbb{D}})$.

Definition 4.4. Let $\tilde{\mathbb{D}}_\alpha = [\tilde{D}_{ij\alpha}]_{N\times M}$, where $\tilde{D}_{ij\alpha}$ is the α-level set of $\tilde{D}_{ij}$. Call the market formed by S and interval matrix $\tilde{\mathbb{D}}_\alpha$ the α-confident sub-market of $\tilde{\mathcal{M}}$, denoted by $\bar{\mathcal{M}}_\alpha = (S, \tilde{\mathbb{D}}_\alpha)$.

Set $\tilde{\mathbb{D}}^L_\alpha = [\tilde{D}^L_{ij\alpha}]_{N\times M}$ and $\tilde{\mathbb{D}}^U_\alpha = [\tilde{D}^U_{ij\alpha}]_{N\times M}$ lower and upper payoff matrices of the fuzzy payoff market with confidence degree α, respectively.

Now we transfer an acceptable market with fuzzy payoff matrix to many acceptable markets with interval payoff matrices. And use the result in Section 3, we can get following equivalent statements about the acceptability.

Proposition 4.2. (1) *Given any $\alpha \in [0, 1]$, $\bar{\mathcal{M}}_\alpha$ is acceptable if and only if the set $\Psi_\alpha = \{\psi \gg 0 | \tilde{\mathbb{D}}^L_\alpha \psi \le S \le \tilde{\mathbb{D}}^U_\alpha \psi\}$ is nonempty.*

(2) *For $\alpha > \beta$, $\Psi_\alpha \subset \Psi_\beta$.*

Proof. The first statement can be got from Proposition 3.2 in Section 3.

For the second statement, note for $\alpha > \beta$,

$$\tilde{\mathbb{D}}^L_\beta \le \tilde{\mathbb{D}}^L_\alpha \le \tilde{\mathbb{D}}^U_\alpha \le \tilde{\mathbb{D}}^U_\beta.$$

So for any $\psi \gg 0$ such that $\tilde{\mathbb{D}}^L_\alpha \psi \le S \le \tilde{\mathbb{D}}^U_\alpha \psi$, we have $\tilde{\mathbb{D}}^L_\beta \psi \le S \le \tilde{\mathbb{D}}^U_\beta \psi$ obviously.

From the proposition, when we need to check the acceptability of a market $\tilde{\mathcal{M}}$ with fuzzy payoff matrix, we only need to check the acceptability of the market with interval payoff matrix, $\bar{\mathcal{M}}_1 = (S, \tilde{\mathbb{D}}_1)$. Even in the simplest triangular setting where all payoffs are triangular fuzzy numbers, we only need to check the reasonability of the crisp market $(S, \mathbb{D}_1)$.

Definition 4.5. Call Ψ_α the acceptable state price set of $\tilde{\mathcal{M}}$ with confidence degree α.

Obviously, Ψ_α has the closure $\bar{\Psi}_\alpha = \{\psi \ge 0 | \tilde{\mathbb{D}}^L_\alpha \psi \le S \le \tilde{\mathbb{D}}^U_\alpha \psi\}$.

4.3. Acceptable prices of a contingent claim with fuzzy random payoff

As before, now we introduce a contingent claim with fuzzy random payoff to the market $\tilde{\mathcal{M}}$. The contingent claim can be represented as a fuzzy vector $\tilde{\mathbb{X}} = (\tilde{X}_1, \ldots, \tilde{X}_M)^{\mathrm{T}}$, where $\tilde{X}_j$ is the fuzzy payoff at state $w = w_j, j = 1, 2, \ldots, M$. Also we call $\tilde{\mathbb{X}}_\alpha = (\tilde{X}_{1\alpha}, \ldots, \tilde{X}_{M\alpha})^{\mathrm{T}}$ the payoff interval vector with confidence degree α, with $\tilde{\mathbb{X}}^L_\alpha = (\tilde{X}^L_{1\alpha}, \ldots, \tilde{X}^L_{M\alpha})^{\mathrm{T}}$ and $\tilde{\mathbb{X}}^U_\alpha = (\tilde{X}^U_{1\alpha}, \ldots, \tilde{X}^U_{M\alpha})^{\mathrm{T}}$ two endpoint vectors of the α-level set. Now under the consideration of acceptability after the introduction of the contingent claim, we want to give some information on the price of $\tilde{\mathbb{X}}$ at date 0. Setting the price of $\tilde{\mathbb{X}}$ be h, denote the market of basic securities and $\tilde{\mathbb{X}}$ be $\hat{\tilde{\mathcal{M}}} = \left(\begin{pmatrix} S \\ h \end{pmatrix}, \begin{pmatrix} \tilde{\mathbb{D}} \\ \tilde{\mathbb{X}}^{\mathrm{T}} \end{pmatrix}\right)$.

Definition 4.6. If h is set such that the market $\hat{\tilde{\mathcal{M}}}_\alpha = \left(\begin{pmatrix} S \\ h \end{pmatrix}, \begin{pmatrix} \tilde{\mathbb{D}}_\alpha \\ \tilde{\mathbb{X}}^{\mathrm{T}}_\alpha \end{pmatrix}\right)$. is acceptable, we call h an acceptable price for $\tilde{\mathbb{X}}$ with confidence degree α.

From results in Section 3, for any $\alpha \in [0, 1]$, define two linear programming:

$$(\mathrm{I})_\alpha \quad \begin{array}{ll} \min & (\tilde{\mathbb{X}}^L_\alpha)^{\mathrm{T}}\psi \\ \text{s.t.} & \psi \in \bar{\Psi}_\alpha \end{array} \qquad (\mathrm{II})_\alpha \quad \begin{array}{ll} \max & (\tilde{\mathbb{X}}^U_\alpha)^{\mathrm{T}}\psi \\ \text{s.t.} & \psi \in \bar{\Psi}_\alpha. \end{array} \tag{19}$$

From Proposition 3.3 in previous section, we can get

Proposition 4.3. *h is an acceptable price for $\tilde{\mathbb{X}}$ with confidence degree α, if and only if $h \in (h^L_\alpha, h^U_\alpha)$, where h^L_α, h^U_α are optimal values of $(\mathrm{I})_\alpha$ and $(\mathrm{II})_\alpha$, respectively.*

Proposition 4.4. *h^L_α, h^U_α are optimal values of following two linear programming $(\mathrm{III})_\alpha$ and $(\mathrm{IV})_\alpha$, respectively.*

$$(\mathrm{III})_\alpha \quad \begin{array}{ll} \max & S^{\mathrm{T}}(\theta_1 - \theta_2) \\ \text{s.t.} & [\tilde{\mathbb{D}}^U_\alpha]^{\mathrm{T}}\theta_1 - [\tilde{\mathbb{D}}^L_\alpha]^{\mathrm{T}}\theta_2 \le \tilde{\mathbb{X}}^L_\alpha \\ & \theta_1, \theta_2 \ge 0 \end{array} \qquad (\mathrm{IV})_\alpha \quad \begin{array}{ll} \min & S^{\mathrm{T}}(\theta_1 - \theta_2) \\ \text{s.t.} & [\tilde{\mathbb{D}}^L_\alpha]^{\mathrm{T}}\theta_1 - [\tilde{\mathbb{D}}^U_\alpha]^{\mathrm{T}}\theta_2 \ge \tilde{\mathbb{X}}^U_\alpha \\ & \theta_1, \theta_2 \ge 0. \end{array} \tag{20}$$

Proof. Rewrite $(\mathrm{I})_\alpha$ and $(\mathrm{II})_\alpha$ as $(\mathrm{I})'_\alpha$ and $(\mathrm{II})'_\alpha$ as follows.

$$(\mathrm{I})'_\alpha \quad \begin{array}{ll} \min & (\tilde{\mathbb{X}}^L_\alpha)^{\mathrm{T}}\psi \\ \text{s.t.} & \tilde{\mathbb{D}}^U_\alpha \psi \ge S \\ & [-\tilde{\mathbb{D}}^L_\alpha]\psi \ge -S \\ & \psi \ge 0 \end{array} \qquad (\mathrm{II})'_\alpha \quad \begin{array}{ll} \max & (\tilde{\mathbb{X}}^U_\alpha)^{\mathrm{T}}\psi \\ \text{s.t.} & \tilde{\mathbb{D}}^U_\alpha \psi \ge S \\ & [-\tilde{\mathbb{D}}^L_\alpha]\psi \ge -S \\ & \psi \ge 0. \end{array} \tag{21}$$

By duality theory on linear programming, we can get the result easily.

Proposition 4.5. (1) *For* $\alpha, \beta \in [0, 1]$, $\alpha > \beta$, $h^L_\beta \le h^L_\alpha \le h^U_\alpha \le h^U_\beta$.

(2) *There is an unique fuzzy price* $\tilde{h}$, *whose* α*-level set is* $[h^L_\alpha, h^U_\alpha]$.

Proof. (1) From Proposition 4.2, we get for any two levels $\alpha > \beta$, $\Psi_\alpha \subset \Psi_\beta$. We also have for $\alpha > \beta$, $\tilde{\mathbb{X}}_\alpha \subset \tilde{\mathbb{X}}_\beta$.

Obviously we have $h^L_\alpha \le h^U_\alpha$ and $h^L_\beta \le h^U_\beta$. Now we need to prove $h^L_\beta \le h^L_\alpha$, and $h^U_\alpha \le h^U_\beta$.

Consider two linear programming:

$$\text{(V)} \quad \begin{array}{ll} \min & (\tilde{\mathbb{X}}^L_\alpha)^{\mathrm{T}}\psi \\ \text{s.t.} & \psi \in \bar{\Psi}_\beta \end{array} \qquad \text{(VI)} \quad \begin{array}{ll} \max & (\tilde{\mathbb{X}}^U_\beta)^{\mathrm{T}}\psi \\ \text{s.t.} & \psi \in \bar{\Psi}_\alpha. \end{array} \tag{22}$$

Denote the optimal values of (V), (VI) be h^* and h^{**} respectively.

From $\Psi_\alpha \subset \Psi_\beta$, we can get $h^* \le h^L_\alpha$ and $h^{**} \le h^U_\beta$. From $\tilde{\mathbb{X}}_\alpha \subset \tilde{\mathbb{X}}_\beta$, or equivalently

$$\tilde{\mathbb{X}}^L_\beta \le \tilde{\mathbb{X}}^L_\alpha \le \tilde{\mathbb{X}}^U_\alpha \le \tilde{\mathbb{X}}^U_\beta,$$

we have $h^* \ge h^L_\beta$ and $h^{**} \ge h^U_\alpha$.

Combining all inequalities together, we can get for $\alpha > \beta$, $h^L_\beta \le h^L_\alpha \le h^U_\alpha \le h^U_\beta$.

(2) We will use Theorem 4.1 to prove the second statement. From (i), h^L_α is nondecreasing and h^U_α is decreasing on α. We also have $h^L_1 \le h^U_1$. Now we want to prove h^L_α and h^U_α are left-continuous and right-continuous at 0.

Set $\alpha_k \uparrow \alpha$, which means $\{\alpha_k\}_{k=1}^\infty$ be a increasing sequence converging to α. We use $(\mathrm{III})_\alpha$ and $(\mathrm{IV})_\alpha$ to prove the result.

Denote

$$\Theta^L_\alpha = \{(\theta_1, \theta_2) | \theta_1, \theta_2 \ge 0, [\tilde{\mathbb{D}}^U_\alpha]^{\mathrm{T}}\theta_1 - [\tilde{\mathbb{D}}^L_\alpha]^{\mathrm{T}}\theta_2 \le \tilde{\mathbb{X}}^L_\alpha\} \tag{23}$$

and

$$\Theta^U_\alpha = \{(\theta_1, \theta_2) | \theta_1, \theta_2 \ge 0, [\tilde{\mathbb{D}}^L_\alpha]^{\mathrm{T}}\theta_1 - [\tilde{\mathbb{D}}^U_\alpha]^{\mathrm{T}}\theta_2 \ge \tilde{\mathbb{X}}^U_\alpha\}. \tag{24}$$

We have

$$\Theta^L_{\alpha_k} \uparrow \Theta^L_\alpha, \qquad \Theta^U_{\alpha_k} \uparrow \Theta^U_\alpha.$$

To see this, for α_k and α as above, from $\tilde{\mathbb{X}}^U_{\alpha_k} \ge \tilde{\mathbb{X}}^U_\alpha$, $\tilde{\mathbb{X}}^U_{\alpha_k} \ge \tilde{\mathbb{X}}^U_\alpha$, $\tilde{\mathbb{X}}^L_{\alpha_k} \le \tilde{\mathbb{X}}^L_\alpha$ and $\theta_1, \theta_2 \ge 0$, we get $\Theta^L_{\alpha_k} \subset \Theta^L_\alpha$. Obviously, for any u, Θ^L_u is a closed convex set. So as $\alpha_k \uparrow \alpha$, we have $\Theta^L_{\alpha_k} \uparrow \Theta^L_\alpha$. Similarly, we can prove $\Theta^U_{\alpha_k} \uparrow \Theta^U_\alpha$.

While $h^L_{\alpha_k}$ and h^L_α are optimal values of the same objective function in $\Theta^L_{\alpha_k}$ and Θ^L_α respectively, we can see that h^L_α is left-continuous. The same argument can prove that h^L_α, h^U_α are left-continuous and right-continuous at 0.

Definition 4.7. Call the fuzzy price $\tilde{h}$ got from above proposition the fuzzy price of $\tilde{\mathbb{X}}$ based on $(S, \tilde{\mathbb{D}})$.

In the market $(S, \tilde{\mathbb{D}}_\alpha)$, we can accept the payoff matrix with confidence degree α. We also accept the price of $\tilde{\mathbb{X}}$ derived from $(S, \tilde{\mathbb{D}}_\alpha)$ with confidence degree α. From the definition, we can see the α-level set of $\tilde{h}$ is just the α-confident acceptable price set.

Definition 4.8. (1) θ is called a sup-replicative strategy for $\tilde{\mathbb{X}}$ in the market $\tilde{\mathcal{M}}$ with confidence degree α, if θ is a super-replicative strategy for $\tilde{\mathcal{M}}_\alpha$ in the market $\bar{\mathcal{M}}_\alpha$. Denote all such sup-replicative strategies as Θ^p_α.

(2) θ is called a sub-replicative strategy for $\tilde{\mathbb{X}}$ in the market $\tilde{\mathcal{M}}$ with confidence degree α, if θ is a super-replicative strategy for $\tilde{\mathcal{M}}_\alpha$ in the market $\bar{\mathcal{M}}_\alpha$. Denote all such sub-replicative strategies as Θ^b_α.

Obviously, we have

$$\Theta^p_\alpha = \{\theta | (\tilde{\mathbb{D}}^{\mathrm{T}}\theta)_\alpha \succeq \tilde{\mathbb{X}}_\alpha\} = \{\theta | (\tilde{\mathbb{D}}^{\mathrm{T}}\theta)^L_\alpha \ge \tilde{\mathbb{X}}^U_\alpha\} \tag{25}$$

$$\Theta^b_\alpha = \{\theta | (\tilde{\mathbb{X}}_\alpha \succeq \tilde{\mathbb{D}}^{\mathrm{T}}\theta)_\alpha\} = \{\theta | (\tilde{\mathbb{D}}^{\mathrm{T}}\theta)^U_\alpha \le \tilde{\mathbb{X}}^L_\alpha\}. \tag{26}$$

Proposition 4.6. (1) *For any* $\theta \in \Theta^p_\alpha$, *there exists* $(\theta_1, \theta_2) \in \Theta^U_\alpha$ *such that* $\theta = \theta_1 - \theta_2$. *And similarly, for any* $\theta \in \Theta^b_\alpha$, *there exists* $(\theta_1, \theta_2) \in \Theta^L_\alpha$ *such that* $\theta = \theta_1 - \theta_2$.

(2) *For* $\alpha > \beta$, $\Theta^p_\alpha \supset \Theta^p_\beta$, $\Theta^b_\alpha \supset \Theta^b_\beta$.

(3) h^L_α *is the maximal value of the sup-replicative strategy for* $\tilde{\mathbb{X}}$ *with confidence degree* α; h^U_α *is the minimal value of the sub-replicative strategy for* $\tilde{\mathbb{X}}$ *with confidence degree* α.

Proof. (1) We can get the result using the same argument as in (17).

(2) For $\alpha > \beta$, from $(\tilde{\mathbb{D}}^{\mathrm{T}}\theta)^L_\alpha \ge (\tilde{\mathbb{D}}^{\mathrm{T}}\theta)^L_\beta$, $(\tilde{\mathbb{D}}^{\mathrm{T}}\theta)^U_\alpha \le (\tilde{\mathbb{D}}^{\mathrm{T}}\theta)^U_\beta$, $\tilde{\mathbb{X}}^U_\alpha \le \tilde{\mathbb{X}}^U_\beta$, $\tilde{\mathbb{X}}^L_\alpha \ge \tilde{\mathbb{X}}^L_\beta$, we can get $\Theta^p_\alpha \supset \Theta^p_\beta$, $\Theta^b_\alpha \supset \Theta^b_\beta$.

Table 1
Fuzzy payoff matrix for two stocks

	w_1	w_2	w_3
Stock A	(26, 26.5, 27.5)	(29.5, 30, 30.5)	(32, 33.5, 35)
Stock B	(24, 24.5, 25.5)	(20, 21, 21.5)	(16, 17, 18)

(3) By Proposition 4.4, the relationship between Θ_α^p and Θ_α^U shows that h_α^U is the optimal value of

$$\begin{aligned}&\min S^{\mathrm{T}}\theta\\ &\text{s.t. } \theta \in \Theta_\alpha^p.\end{aligned} \tag{27}$$

which confirm the first statement. Similarly, we can see that h_α^L is the optimal value of

$$\begin{aligned}&\max S^{\mathrm{T}}\theta\\ &\text{s.t. } \theta \in \Theta_\alpha^b.\end{aligned} \tag{28}$$

Now we can conclude the pricing result for the contingent claim with fuzzy random payoff. The claim will have a fuzzy price. For any level in its α-level set, $[\tilde{h}_\alpha^L, \tilde{h}_\alpha^U]$, we can accept it with confidence degree α. And the two endpoints of the set can be explained as the minimal value of α-confident sup-replicative strategies and the maximal value of α-confident sub-replicative strategies, respectively. Also noticing that a random interval is a special fuzzy random variable, the interval price obtained in random interval setting is a special fuzzy price obtained in fuzzy random setting.

4.4. A procedure for computation and an example

When we use models to get the price for a contingent claim with fuzzy random payoff, the most important factors are α-level sets for any entries. A useful way to get the level set are from the membership function. In application, we can use following procedure.

Procedure for get the fuzzy price for a contingent claim with fuzzy random payoff

(i) Forecast the state set Ω at date 1.
(ii) Get price-vector S and information for fuzzy random payoff of basic securities from historical data.
(iii) Translate the information into the fuzzy payoff matrix $\mathbb{D}$.
(iv) Get membership functions of all fuzzy payoffs.
(v) For any $\alpha \in [0, 1]$, get the interval-valued payoff matrix $\tilde{\mathbb{D}}_\alpha$ with confidence degree α.
(vi) Check the acceptability of the fuzzy market $\tilde{M} = (S, \tilde{\mathbb{D}})$.
(vii) Get the payoff vector of a contingent claim $\tilde{\mathbb{X}}$ and membership functions for entries of $\tilde{\mathbb{X}}$.
(viii) Compute h_α^L and h_α^U to form α-level set of its fuzzy price.
(ix) Combine level sets into a fuzzy price.

Example. A week later, the price for crude oil will be adjusted. Five states are estimated to happen. The state set is $\Omega = \{w_1, w_2, w_3\}$, where $w_1 =$ "decreasing by 1%–2%", $w_2 =$ "unadjusted" and $w_3 =$ "increasing by 1%–2%". There are two stocks: Stock A in petroleum industry, Stock B in airline industry. The price vector for three stocks is $S = (\$30, \$20)$. From history data, we can forecast the payoff of every stock in every state, all modeled by triangle fuzzy numbers for simplicity, listed in Table 1. A triangle fuzzy number is denoted as $\tilde{a} = (a_1, a_2, a_3)$, $a_1 \le a_2 \le a_3$ with the membership function

$$\mu_{\tilde{a}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} a_1 \le x \le a_2 \\ \dfrac{x - a_3}{a_2 - a_3} a_2 \le x \le a_3 \\ 0 \quad \text{else} \end{cases}$$

and the α-level set of $\tilde{a}$ is $[a_1 + \alpha(a_2 - a_1), a_3 + \alpha(a_2 - a_3)]$.

Given confidence degree $\alpha \in [0, 1]$, $\tilde{\mathbb{D}}_\alpha^L$, $\tilde{\mathbb{D}}_\alpha^U$ can be given as

$$\tilde{\mathbb{D}}_\alpha^L = \begin{bmatrix} 26 + 0.5\alpha & 29.5 + 0.5\alpha & 32 + 0.5\alpha \\ 24 + 0.5\alpha & 20 + \alpha & 16 + \alpha \end{bmatrix}$$

$$\tilde{\mathbb{D}}_\alpha^U = \begin{bmatrix} 27.5 - \alpha & 30.5 - 0.5\alpha & 35 - 1.5\alpha \\ 25.5 - \alpha & 21.5 - 0.5\alpha & 18 - \alpha \end{bmatrix}.$$

On the acceptability of the fuzzy payoff market, from Proposition 4.2, we only need to check the reasonability of the crisp market $(S, \tilde{\mathbb{D}}_1)$, where $\tilde{\mathbb{D}}_1 = \begin{bmatrix} 26.5 & 30 & 33.5 \\ 24.5 & 21 & 17 \end{bmatrix}$. For this crisp market, we can easily get a state price vector (320/1481, 160/387, 1073/3028), which shows that there are no arbitrages in the market $(S, \tilde{\mathbb{D}}_1)$.

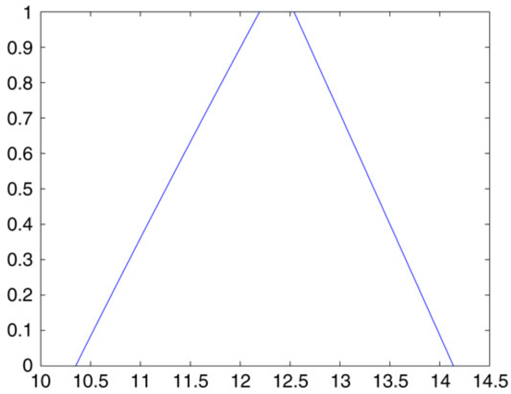

Fig. 2. Membership function of the fuzzy price for the contingent claim.

Now there is a contingent claim, $\tilde{\mathbb{X}}$, with payoff $\tilde{\mathbb{X}} = ((8, 9, 10), (11, 12, 13), (14, 15, 16))^{\mathrm{T}}$. We can get the α-level set of $\tilde{\mathbb{X}}$ be

$$\tilde{\mathbb{X}}_\alpha = ([8+\alpha, 10-\alpha], [11+\alpha, 13-\alpha], [14+\alpha, 16-\alpha])^{\mathrm{T}}.$$

For h_α^L, h_α^U, we should solve following two parametric linear programming:

$$\begin{aligned} h_\alpha^L = \min \quad & (8+\alpha)\psi_1 + (11+\alpha)\psi_2 + (14+\alpha)\psi_3 \\ \text{s.t.} \quad & (26+0.5\alpha)\psi_1 + (29.5+0.5\alpha)\psi_2 + (32+0.5\alpha)\psi_3 \le 30 \\ & (24+0.5\alpha)\psi_1 + (20+\alpha)\psi_2 + (16+\alpha)\psi_3 \le 20 \\ & (27.5-\alpha)\psi_1 + (30.5-0.5\alpha)\psi_2 + (35-1.5\alpha)\psi_3 \ge 30 \\ & (25.5-\alpha)\psi_1 + (21.5-0.5\alpha)\psi_2 + (18-\alpha)\psi_3 \ge 20 \\ & \psi_1, \psi_2, \psi_3 \ge 0 \end{aligned}$$

$$\begin{aligned} h_\alpha^U = \max \quad & (10-\alpha)\psi_1 + (13-\alpha)\psi_2 + (16-\alpha)\psi_3 \\ \text{s.t.} \quad & (26+0.5\alpha)\psi_1 + (29.5+0.5\alpha)\psi_2 + (32+0.5\alpha)\psi_3 \le 30 \\ & (24+0.5\alpha)\psi_1 + (20+\alpha)\psi_2 + (16+\alpha)\psi_3 \le 20 \\ & (27.5-\alpha)\psi_1 + (30.5-0.5\alpha)\psi_2 + (35-1.5\alpha)\psi_3 \ge 30 \\ & (25.5-\alpha)\psi_1 + (21.5-0.5\alpha)\psi_2 + (18-\alpha)\psi_3 \ge 20 \\ & \psi_1, \psi_2, \psi_3 \ge 0. \end{aligned}$$

Letting α taking values from 0 to 1 with step 0.01, we can get the fuzzy price $\tilde{h}$, with its membership function depicted in Fig. 2.

Remarks to the result of above example is given as follows. First, the price got is a fuzzy number, which confirms the pricing propositions. Second, the resulting price is trapezoidal, while not triangular. If we check the crisp market as $\alpha = 1$, where $\tilde{\mathbb{D}}_1 = \begin{bmatrix} 26.5 & 30 & 33.5 \\ 24.5 & 21 & 17 \end{bmatrix}$, the market $(S, \tilde{\mathbb{D}}_1)$ is obviously incomplete, because there are two basic securities and three states. Also, we can see $\tilde{\mathbb{X}}_1 = (9, 12, 15)^{\mathrm{T}}$ can't be replicated by two basic securities. By the classical argument, as $\alpha = 1$, the contingent claim has a price interval, which is just the corresponding level cut of the fuzzy price.

5. Conclusion and further discussion

This article proposes a novel pricing method in the market with random interval and fuzzy random payoffs in single-period. Using theories on uncertain analysis, including interval number theory, fuzzy theory and fuzzy random theory combined with uncertain programming theories, we expand the discussion according to classical discussions in random single period market. Following classical no-arbitrage principle, we give a new concept of acceptable market. Under the view of acceptability, we get some interesting results using acceptable state price vectors and novel replicative arguments. In contrast to classical results, a unique price or a price interval can't be derived. We get an price interval in random interval setting and a fuzzy price in fuzzy random setting. After the introduction of replicative strategies, some familiar results are got:

(a) In random interval setting, an acceptable price interval is got, with two endpoints explained as the maximal value of sub-replicative strategies and the minimal value of sup-replicative strategies for the contingent claim.
(b) In fuzzy random setting, a fuzzy price is obtained. And for any α-level set of the fuzzy price, two endpoints can also be explained as the maximal value of sub-replicative strategies and the minimal value of sup-replicative strategies with confidence degree α for the contingent claim.

The authors want to encourage discussion and further research into the market with fuzzy or other uncertain factors. Here are some open questions even in one-period setting. Can we introduce a risk-free security? How can we repress the result using some concept like martingale in random financial theory? Can we extend the proposed discussion to multi-period setting and then to continuous setting? Can we discuss other problems such as utility maximization and market optimality in fuzzy random market? All these problem needs further discussion on fuzzy random theory and optimization theory. We can be sure that all investor will benefit if we can extend the result in the classical random financial theory to broader uncertain areas, such as pricing theory, risk management tools and portfolio selection analysis.

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